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Some Spectral Properties of Weighted Random Processes

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Abstract

We study the power spectrum and, more generally, the spectral covariance of weighted stationary processes. It is found that if the power spectrum of the underlying stationary process is suitably well behaved and properly matched to the weight function, then the high-frequency behavior of the power spectrum and spectral covariance is especially simple. Asymptotic theorems describing this behavior precisely are given.

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1. Introduction

In many problems involving random noise, the object of primary interest is a stationary random process, $x(t)$ say, but what is actually available for observation instead is the weighted process $y(t) = m(t)x(t)$, obtained by multiplying $x(t)$ by the non-random weight function $m(t)$. For example, in analyzing a radio scattering experiment, the dielectric noise $x(\vec{r})$*, which is usually regarded as spatially stationary, is multiplied by a weight function $m(\vec{r})$ representing the joint effect of the gain patterns of the transmitting and receiving antennas, and only the region where $m(\vec{r})$ is large, the so-called scattering volume, makes an appreciable contribution to the received scattered signal[1]. Further examples abound in the important problem of experimental power spectrum measurement, where the weight functions $m(t)$ correspond to the <u>data windows</u> discussed by Blackman and Tukey[2]; these authors give a detailed analysis of the practical consequences of various choices of $m(t)$. Weighted processes have also been studied by Pugachev (see [3], p. 330) and Burford[4]. Indeed, in a sense, weighted processes are more physical than stationary processes, since the latter have the unreasonable property of continuing forever with constant average noise power. (Of course, this property is never taken too seriously by applied people, who customarily allude to stationary processes as convenient idealizations of physical noises).

The question with which we are mainly concerned in this paper is roughly the following: Under what conditions will the power spectra of $x(t)$ and $y(t)$ be approximately the same for sufficiently high frequencies?

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* By the dielectric noise we mean the refractive index fluctuations usually attributed to the action of atmospheric turbulence. Here we replace the scalar t (time) by the vector $\vec{r}$ (position), since we are dealing with a random field.

(For, whereas it is to be expected that in general $x(t)$ and $y(t)$ will have quite different power spectra at low frequencies, because of the modulatory action of $m(t)$, it is quite possible for the spectra to be nearly the same at sufficiently high frequencies.) We find that to give even a partial answer to this question we must explore the asymptotic behavior of convolutions.

2. Preliminary analysis

Let $x(t)$ be a zero-mean stationary process, so that $Ex(t) = 0$, $Ex(t)\overline{x}(t') = C(t-t')$, where as usual E denotes the expectation or ensemble average, and the overbar denotes the complex conjugate. We assume that the autocorrelation function $C(\tau)$ is continuous and absolutely integrable, and write the Wiener-Khintchine relations in the form

$$C(\tau) = \int_{-\infty}^{\infty} \exp(i\omega\tau) f(\omega) d\omega \quad ,$$

$$f(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-i\omega\tau) C(\tau) d\tau \quad ,$$

where the power spectrum (more accurately, the power spectral density) $f(\omega)$ is non-negative, continuous and integrable; if $x(t)$ is real, $f(\omega)$ is even, but more generally we allow $x(t)$ to be complex. Now let $m(t)$ be a bounded continuous function, and construct the weighted process $y(t) = m(t)x(t)$. Clearly $Ey(t) = 0$, $Ey(t)\overline{y}(t') = m(t)\overline{m}(t')C(t-t')$, so that in particular $y(t)$ is not stationary. (However, the normalized process $y(t)/\left[E|y(t)|^2\right]^{1/2}$ is stationary.) We assume that $m(t)$ is square-integrable, which implies that the sample functions of $y(t)$, unlike those of $x(t)$, are themselves square-integrable with probability one. To see this, note that

$$E\int_{-\infty}^{\infty} |y(t)|^2 dt = \int_{-\infty}^{\infty} E|y(t)|^2 dt = C(0) \int_{-\infty}^{\infty} |m(t)|^2 dt \quad < \infty \ .$$

We can therefore take (L^2) Fourier transforms of the sample functions of

y(t) individually, obtaining the spectral process*

$$(1) \qquad Y(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-i\omega t)y(t)dt$$

with (spectral)covariance $\Gamma(\omega,\omega') = Ey(\omega)\bar{Y}(\omega')$.

We now find the relation between $\Gamma(\omega,\omega')$ and the power spectrum of x(t). First we observe that

$$(2) \qquad x(t) = \int_{-\infty}^{\infty} \exp(i\omega t)dX(\omega) \quad ,$$

where the integral is meant in the mean square sense, and the increments of the spectral process $X(\omega)$ obey the symoblic relation**

$$(3) \qquad EdX(\omega)\overline{dX}(\omega') = f(\omega)\delta(\omega-\omega')d\omega d\omega' \quad .$$

It follows from (1) and (2) that

$$Y(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-i\omega t)m(t)x(t)dt = \int_{-\infty}^{\infty} M(\omega-\omega')dX(\omega') \quad ,$$

where

$$M(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-i\omega t)m(t)dt$$

is the Fourier transform of the weight function m(t). By Plancherel's theorem $M(\omega)$ is itself square-integrable, and in fact $\int_{-\infty}^{\infty} |M(\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |m(t)|^2 dt$. Finally, forming the covariance of $Y(\omega)$ and using (3) we obtain

$$(4) \qquad \Gamma(\omega,\omega') = \int_{-\infty}^{\infty} M(\omega-s)\bar{M}(\omega'-s)f(s)ds \quad ,$$

the desired relation between $\Gamma(\omega,\omega')$ and $f(\omega)$. The quantity $\Gamma(\omega,\omega) = E|Y(\omega)|^2$ is appropriately called the power spectrum of y(t), even though y(t) is not stationary; for simplicity we shall henceforth abbreviate $\Gamma(\omega,\omega')$ to just $\gamma(\omega)$. Specializing (4) to the case $\omega = \omega'$, we obtain

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* Here, and occasionally below, the integral means $\underset{A\to\infty}{\text{l.i.m.}} \int_{-A}^{A}$.

** See e.g. [5], ch. 10; $\delta(\omega)$ is the Dirac delta function.

$$(5) \qquad \gamma(\omega) = \int_{-\infty}^{\infty} K(\omega-s)f(s)ds = \int_{-\infty}^{\infty} K(s)f(\omega-s)ds \qquad ,$$

where we have introduced the notation $K(\omega) \equiv |M(\omega)|^2$. Thus, the power spectrum of $y(t)$ is the convolution of the power spectrum of $x(t)$ with the non-negative integrable kernel $K(\omega)$. This is a familiar result, and is given e.g. in [2]. In what follows, we shall assume, as we may without loss of generality, that $\int_{-\infty}^{\infty} K(\omega)d\omega = 1$; otherwise, we need only supply an appropriate constant factor in certain obvious places.

3. Asymptotic behavior of $\gamma(\omega)$

Examining the convolution (5), we expect that if the kernel $K(\omega)$ is small compared to $f(\omega)$ as $\omega \to \infty$, and if $f(\omega)$ does not vary too rapidly, then $\gamma(\omega)$ should behave like $f(\omega)$ for large ω, since $\gamma(\omega)$ is an average of $f(s)$ with weight $K(\omega-s)$ centered at $s = \omega$. (If, on the other hand, $K(\omega)$ is *large* compared to $f(\omega)$, the roles of $f(\omega)$ and $K(\omega)$ are interchanged.) Speaking even more qualitatively, when two functions are convolved it should often be possible to regard the function which falls off more rapidly as averaging the function which falls off less rapidly. We shall see that this intuitive idea can be made precise under fairly general hypotheses, but that it breaks down when $f(\omega)$ is too rapidly decreasing. First we must specify what we mean by a function which does not vary too rapidly.

Definition. A function $f(\omega)$ *is slowly varying as* $\omega \to \infty$ *(briefly, slowly varying) if* $f(\omega)/f(\sigma)$ *tends uniformly to* 1 *whenever* $\omega, \sigma \to \infty$ *in such a way that* $\omega/\sigma \to 1$.

Slowly varying functions are related to the *slowly oscillating* functions that arise in connection with Tauberian theorems. A slowly oscillating

function $g(\omega)$ is one for which $|g(\omega)-g(\sigma)| \to 0$ uniformly whenever $\omega, \sigma \to \infty$ in such a way that $\omega/\sigma \to 1$*; a slowly varying function as defined above is just the exponential of a slowly oscillating function. For example, the functions $(1 + |\omega|^p)^{-1}$ and $(1 + |\omega|^p)^{-1}\log(1 + |\omega|)$, $p > 0$, are slowly varying, but $\exp(-|\omega|^p)$, $p > 0$, is not. The following facts about slowly varying functions are an immediate consequence of the definition.

1. If $f(\omega)$ is slowly varying, it does not vanish for sufficiently large ω; this observation is important in connection with certain expressions written below where $f(\omega)$ appears in the denominator.

2. The product and quotient of two slowly varying functions are slowly varying.

3. If for large ω the function $g(\omega)$ has a derivative and if $|g'(\omega)| \leq C/\omega$, then $|g(\omega)-g(\sigma)| = \left| \int_\sigma^\omega g'(s)ds \right| \leq C(\frac{\omega}{\sigma} - 1)$, so that $g(\omega)$ is slowly oscillating. Thus, for $f(\omega)$ to be slowly varying it is sufficient that $f(\omega)$ be differentiable and non-vanishing and that $\omega \frac{d}{d\omega} \log f(\omega)$ be bounded, all for sufficiently large ω.

4. If $f(\omega)$ is slowly varying, positive and continuous, then if $q > 0$ is sufficiently large we have $f(\omega) < \omega^q$ for large ω. The proof is as follows: for $\omega > \omega_o$, $f(\omega)/f(\rho\omega) \leq 2$, if ρ is sufficiently close to 1 $(0 < \rho < 1)$. Iterating this inequality n times, where n is the largest integer such that $\rho^{n-1}\omega > \omega_o$, we obtain $f(\omega) \leq 2^n f(\rho^n\omega) \leq M2^n$, where M is the maximum of $f(\omega)$ in the interval $(\rho\omega_o, \omega_o)$. Now $2^n = \rho^{-cn} < (\rho\omega_o/\omega)^{-c} = A\omega^c$, where $\rho^{-c} = 2$, so that $f(\omega) \leq MA\omega^c$, which proves the result. Moreover, since $1/f(\omega)$ is slowly varying, we also have $f(\omega) > \omega^{-p}$ for suitable $p > 0$ and large ω.

It follows from the last remark that slowly varying functions are, roughly speaking, functions which behave like a power of ω at infinity. It

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* See e.g. [6], p. 124.

will be noted that the class of slowly varying power spectra is comfortably large: for example, it contains all spectra obtained by passing white noise through linear networks, but not band-limited spectra. (However, band-limited spectra are physically unrealizable.)

Returning to our heuristic notions, we require that the kernel $K(\omega)$ be small compared to $f(\omega)$ for large ω. The mathematical condition which expresses this most naturally is that

$$(6) \qquad \lim_{\omega \to \infty} K(\omega)/f(\omega) = 0 \quad .$$

It is important to note that (6) implies nothing about the relative "widths" of $f(\omega)$ and $K(\omega)$. For example, the "halfwidth" of $K(\omega) = T/\pi\left[1+(T\omega)^2\right]$ is $1/T$, while that of $f(\omega) = 1/(1+|\omega|^p)$, $0 < p < 2$, is 1, but with this choice (6) is valid for <u>any</u> $T > 0$. Thus, no restriction on the width of data windows will figure in our results.

We are now in a position to state

<u>Theorem 1.</u> (<u>Asymptotic convolution theorem</u>) <u>Let $f(\omega)$ and $K(\omega)$ be non-negative integrable functions (with $\int_{-\infty}^{\infty} K(\omega)d\omega = 1$). If $f(\omega)$ is slowly varying and non-increasing for sufficiently large ω, and if</u>

$$(6) \qquad \lim_{\omega \to \infty} K(\omega)/f(\omega) = 0 \quad ,$$

<u>then</u>

$$(7) \qquad \lim_{\omega \to \infty} \gamma(\omega)/f(\omega) = 1 \quad ,$$

<u>where $\gamma(\omega)$ is the convolution</u> (5).

Remark 1. Of course, the theorem remains true if ∞ is replaced by $-\infty$ in the hypotheses and conclusions.

Remark 2. Actually, we need not assume that $f(\omega)$ is non-increasing, but the somewhat milder restriction

(8) $$f(\omega) \leq Cf(\sigma) \quad , \text{ when } \omega_o < \omega < \sigma \quad ,$$

where C is a constant independent of ω. It seems plausible that (8) is actually a redundant hypothesis, i.e. that $f(\omega)$ being integrable and slowly varying implies (8), but we could not show this.

Proof of Theorem 1. We have

(9) $$\gamma(\omega) = \int_{-\infty}^{R} K(s)f(\omega-s)ds + \int_{R}^{\infty} K(s)f(\omega-s)ds \quad ,$$

where $R = R(\omega)$ is a function of ω to be specified later. For the moment we suppose only that

(10) $$R(\omega) \to \infty \text{ as } \omega \to \infty \quad ,$$

(11) $$\lim_{\omega\to\infty} R(\omega)/\omega = 0 \quad .$$

Let us denote the two integrals in (9) by $\gamma_1(\omega)$ and $\gamma_2(\omega)$, respectively. We consider first the quantity

$$\gamma_1(\omega)/f(\omega) = \int_{-\infty}^{R} K(s)\left[f(\omega-s)/f(\omega)\right]ds \quad ,$$

which it is convenient to rewrite as

(12) $$\gamma_1(\omega)/f(\omega) = \int_{-\infty}^{\infty} K(s)\phi_R(s)\left[f(\omega-s)/f(\omega)\right]ds \quad ,$$

where $\phi_R(s)$ is the characteristic function of the interval $(-\infty, R(\omega))$, i.e. the function equal to 1 when s lies in $(-\infty, R(\omega))$ and to 0 otherwise. Now the ratio $f(\omega-s)/f(\omega)$ is bounded in the two-dimensional region $\omega > \omega_o, -\infty < s < R(\omega)$, where ω_o is a suitably large number. For $0 \leq s < R(\omega)$, this follows from (11) and the fact that $f(\omega)$ is slowly

varying; for $s < 0$, it follows from the fact that $f(\omega)$ is non-increasing (or, more generally, satisfies (8)). Moreover, for each fixed s, $\lim_{\omega \to \infty} f(\omega - s)/f(\omega) = 1$ (again because $f(\omega)$ is slowly varying). Thus since $K(\omega)$ is integrable, we can use the Lebesgue dominated convergence theorem to justify passing to the limit under the integral in (12), with the result

$$\lim_{\omega \to \infty} \gamma_1(\omega)/f(\omega) = \int_{-\infty}^{\infty} K(s)ds = 1 \quad .$$

Consequently, the proof of Theorem 1 will be complete when we show that

$$\lim_{\omega \to \infty} \gamma_2(\omega)/f(\omega) = 0 \quad . \tag{13}$$

Now, in view of (6) we can write

$$K(\omega) = \varepsilon(\omega) f(\omega) \quad ,$$

where $\mathcal{E}(\omega) \to 0$ as $\omega \to \infty$. This gives

$$\gamma_2(\omega) = \int_R^{\infty} \varepsilon(s) f(s) f(\omega - s) ds$$

for the second integral in (9). Therefore, since $f(\omega)$ is non-increasing (or satisfies (8))

$$\gamma_2(\omega) \leq \varepsilon_0(R) C f(R) \int_{-\infty}^{\infty} f(\omega - s) ds = C' \varepsilon_0(R) f(R) \quad ,$$

where C' is a constant and

$$\varepsilon_0(\omega) = \sup_{\sigma \geq \omega} \varepsilon(\sigma) \quad .$$

Note that $\varepsilon_0(\omega)$ is non-increasing and $\to 0$ as $\omega \to \infty$. Hence, to establish (13) and thereby complete the proof, it suffices to construct a function $R(\omega) \to \infty$ which satisfies (11) and is such that

$$\lim_{\omega \to \infty} \varepsilon_0(R) f(R)/f(\omega) = 0 \quad . \tag{14}$$

Since $f(\omega)$ is slowly varying, there exists a number ρ, $0 < \rho < 1$, such that

(15) $$f(\rho\omega) \leq 2f(\omega) \quad ,$$

provided that ω is suitably large. Iteration of (15) leads to the relation

(16) $$f(\rho^n\omega) \leq 2^n f(\omega) \quad ,$$

which holds for all n, again provided ω is in each case suitably large. Now, since $\mathcal{E}_o(\omega) \to 0$, we can construct a sequence $\omega_1 < \omega_2 < \omega_3 < \ldots$ of positive numbers with the properties

(17) $$\omega_n > n\rho^{-n} \quad , \qquad \mathcal{E}_o(\rho^n\omega_n) < 3^{-n} \quad ,$$

and also large enough to guarantee the validity of (16) for $\omega \geq \omega_n$. We now define

$$R(\omega) = \rho^n\omega \quad , \quad \omega_n \leq \omega < \omega_{n+1} \quad .$$

Since $R(\omega) \geq \rho^n\omega_n > n$, for $\omega_n \leq \omega < \omega_{n+1}$, (10) is satisfied. Moreover, since $R(\omega)/\omega = \rho^n$ for $\omega_n \leq \omega < \omega_{n+1}$, (11) holds. Finally, using (16) and (17) we have

$$\mathcal{E}_o(R)f(R)/f(\omega) \leq 2^n\, \mathcal{E}_o(\rho^n\omega_n) < (2/3)^n \quad ,$$

for $\omega_n \leq \omega < \omega_{n+1}$, which establishes (14) and completes the proof of Theorem 1.

4. Examples and extensions

We now give examples which illustrate the meaning of Theorem 1. In some examples one or the other of the hypotheses of the theorem is violated, with the result that $\lim_{\omega\to\infty} \gamma(\omega)/f(\omega) \neq 1$. In other examples the requirements on $f(\omega)$ are weakened but those on $K(\omega)$ are

strengthened, with the result that $\lim_{\omega\to\infty} \gamma(\omega)/f(\omega) = 1$ remain valid.

1. An especially simple weight function is

$$m(t) = \sqrt{\pi/T} \qquad , \quad |t| \leq T \quad ,$$

$$m(t) = 0 \qquad , \quad |t| > T \quad ,$$

i.e. a "wide open" data window. The corresponding (Fejér) kernel is

$$K(\omega) = M^2(\omega) = \sin^2 T\omega/\pi T\omega^2 \quad .$$

If now $f(\omega) = 1/(1+|\omega|^p)$, $1 < p < 2$, then according to Theorem 1, no matter how close p is to 2 and regardless of the size of T, the convolution (5) approximates $f(\omega)$ for sufficiently large ω. (How large ω must be for good approximation will depend, of course, on p and T.) This result would be difficult to verify by direct calculation.

2. Even when it can be done, it is usually very tedious to verify Theorem 1 directly. For example, let $K(\omega) = \sqrt{2}/\pi(1+\omega^4)$, $f(\omega) = 1/(1+\omega^2)$, which satisfy the hypotheses of Theorem 1. The convolution of these two functions can be evaluated by residues and is found to be

$$\gamma(\omega) = (\omega^6+6\omega^4+2\omega^2)/(\omega^8+4\omega^6+8\omega^4-8\omega^2+4) + \sqrt{2}(\omega^2-6)/(\omega^6+4\omega^4+4\omega^2+16) \quad .$$

We see at once that $\lim_{\omega\to\infty} \gamma(\omega)/f(\omega) = 1$.

3. As an example of a case where condition (6) is violated, choose $K(\omega)$ and $f(\omega)$ to be the same (slowly varying) function $1/\pi(1+\omega^2)$. Then the convolution (5) is found to be $\gamma(\omega) = 2/\pi(\omega^2+4)$, so that $\lim_{\omega\to\infty} \gamma(\omega)/f(\omega) = 2$ instead of 1.

4. Choose $f(\omega) = e^{-a|\omega|}$, $a > 0$, which is not slowly varying, and let $K(\omega)$ be decreasing for large ω and such that $\int_{-\infty}^{\infty} e^{a\omega}K(\omega)d\omega = A < \infty$. The the convolution (5) is

$$\gamma(\omega) = e^{-a\omega}\int_{-\infty}^{\omega} e^{as}K(s)ds + e^{a\omega}\int_{\omega}^{\infty} e^{-as}K(s)ds \quad .$$

Since for large enough ω the second integral does not exceed $e^{a\omega}K(\omega)\int_{\omega}^{\infty} e^{-as}ds = K(\omega)/a$, and since $\lim_{\omega\to\infty} K(\omega)/f(\omega) = 0$, it follows that $\lim_{\omega\to\infty} \gamma(\omega)/f(\omega) = A$, where in general $A \neq 1$.

5. Choose $f(\omega) = |\omega|e^{-|\omega|}$ and $K(\omega) = \frac{1}{2}e^{-|\omega|}$. Although $f(\omega)$ is not slowly varying, (6) is satisfied. The convolution (5) is an elementary integral and is found to be $\gamma(\omega) = \frac{1}{4}e^{-\omega}(1+\omega+\omega^2)$, $\omega > 0$, so that $\lim_{\omega\to\infty} \gamma(\omega)/f(\omega) = \infty$.

6. As a more complicated example, choose $K(\omega) = ae^{-|\omega|^p}$, $p > 0$*, and $f(\omega) = e^{-|\omega|^q}$, where $0 < q < p/(p+1)$. Again $f(\omega)$ is not slowly varying, but (6) holds, and it is in fact true that $\lim_{\omega\to\infty} \gamma(\omega)/f(\omega) = 1$. To show this, we note first that in the proof of Theorem 1 we can replace the hypothesis that $f(\omega)$ be slowly varying by the weaker condition

(18) $\qquad f(\omega)/f(\sigma) \to 1$ uniformly as $\omega,\sigma\to\infty$ in such a way that $|\omega-\sigma| < R(\omega)$

in showing that $\lim_{\omega\to\infty} \gamma_1(\omega)/f(\omega) = 1$, for the full force of the slowly varying condition is not needed in this part of the proof. If now we strengthen the requirements on $K(\omega)$ to

$$K(\omega) \text{ is non-increasing for large enough } \omega \quad ,$$

(19)
$$\lim_{\omega\to\infty} K(R)/f(\omega) = 0 \quad ,$$

then, since $\gamma_2(\omega) = \int_R^{\infty} K(s)f(\omega-s)ds$ is bounded above by $K(R)\int_{-\infty}^{\infty} f(\omega-s)ds$, $\lim_{\omega\to\infty} \gamma_2(\omega)/f(\omega) = 0$ by (19), so that $\lim_{\omega\to\infty} \gamma(\omega)/f(\omega) = 1$ is still valid. Now let $R(\omega) = \omega^{\alpha}$, $0 < \alpha < 1$, where α will be chosen later. Then, if

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* The constant a is chosen to make $\int_{-\infty}^{\infty} K(\omega)d\omega = 1$.

$0 < \omega < \sigma$, we have

$$f(\omega)/f(\sigma) = \exp(\sigma^q - \omega^q) = \exp\left[q(\sigma-\omega)\rho^{q-1}\right] , \quad \omega < \rho < \sigma ,$$

by the mean value theorem. If $\sigma - \omega < \omega^a$, the exponent on the right will tend to zero, i.e. (18) will hold if $\alpha + q < 1$. Moreover, $\lim_{\omega\to\infty} K(\omega^\alpha)/f(\omega) = 0$, provided that we have $q < p\alpha$ as well as $\alpha + q < 1$. These inequalities are compatible if we can insert a number between q/p and $1-q$, i.e. if $q/p < 1-q$ or $q < p/(p+1)$, which completes the demonstration of our example.

7. The conclusion of Theorem 1 remains true if we replace the hypothesis that $f(\omega)$ is integrable by the weaker hypothesis that $f(\omega)$ is <u>bounded</u>, provided that we also replace (6) by the stronger condition

$$\lim_{\omega\to\infty} \int_\omega^\infty K(s)ds \Big/ f(\omega) = 0.$$

For, since integrability of $f(\omega)$ was used only in showing that $\gamma_2(\omega)$ satisfies $\lim_{\omega\to\infty} \gamma_2(\omega)/f(\omega) = 0$, it suffices to show that the same is true with the altered hypotheses. Let us define $\mathcal{E}(\omega) = \int_\omega^\infty K(s)ds \Big/ f(\omega)$, which by hypothesis $\to 0$ as $\omega \to \infty$. Then if $f(\omega) \leq M$

$$\gamma_2(\omega)/f(\omega) \leq M \int_R^\infty K(s)ds/f(\omega) \leq M\varepsilon(R)f(R)/f(\omega) ,$$

and as above we know that we can construct a function $R = R(\omega)$ satisfying (10) and (11) and such that $\lim_{\omega\to\infty} \mathcal{E}(R)f(R)/f(\omega) = 0$. Hence the proof is complete. Applying this result to the kernel of example 1, we have $\int_\omega^\infty K(s)ds = O(1/\omega)$, so that the example is valid for $0 \leq p < 1$ as well as for $1 < p < 2$.

8. Continuing our discussion of example 1, we note that the case

$p = 1$ is not covered by the arguments given so far. This case may be inferred from the following extension of Theorem 1: if $f(\omega)$ is assumed to be of class $L^{q'}$, $q' > 1$, rather than of class L^1, and we define $q = q'/(q'-1)$, then the conclusion of Theorem 1 remains valid if (6) is replaced by the requirement

$$\lim_{\omega\to\infty}\left\{\int_{\omega}^{\infty}[K(s)]^q ds\right\}^{1/q} \Big/ f(\omega) = 0.$$

For we have only to apply Hölder's inequality, obtaining

$$\gamma_2(\omega) = \int_R^{\infty} f(\omega-s)K(s)ds \leq \left\{\int_R^{\infty}[f(\omega-s)]^{q'}ds\right\}^{1/q'}\left\{\int_R^{\infty}[K(s)]^q ds\right\}^{1/q}$$

$$\leq C\left\{\int_R^{\infty}[K(s)]^q ds\right\}^{1/q} \quad ,$$

and then proceed as in the previous example. In particular, if $K(\omega) = O(\omega^{-m})$ with $m > 1$, and $f(\omega) = 1/(1+|\omega|^p)$, $0 < p \leq 1$, choose $q' = \mu/p$, where $\mu > 1$ is to be determined. Then $f(\omega) \in L^{q'}$ and we have only to verify that $(\omega^{-qm+1})^{1/q}\omega^p \to 0$ as $\omega \to \infty$, i.e. that the exponent $-m + \frac{1}{q} + p = p + 1 - \frac{p}{\mu} - m$ is negative, which is clearly true when μ is close enough to 1. Hence, combining results, we find that example 1 is valid for $0 \leq p < 2$.

5. Asymptotic behavior of $\Gamma(\omega,\omega')$

So far we have been concerned with the power spectrum $\gamma(\omega) = E|Y(\omega)|^2$. We now turn our attention to the covariance $\Gamma(\omega,\omega') = EY(\omega)\bar{Y}(\omega')$, given by

$$(4) \qquad \Gamma(\omega,\omega') = \int_{-\infty}^{\infty} M(\omega-s)\bar{M}(\omega'-s)f(s)ds \quad .$$

The asymptotic behavior of $\Gamma(\omega,\omega')$ is described by the following

generalization of Theorem 1:

Theorem 2. Let $M(\omega)$ be square-integrable with autoconvolution

$$\mathcal{m}(\omega) = \int_{-\infty}^{\infty} M(\omega+s)\overline{M}(s)ds \quad , \quad \mathcal{m}(0) = 1 \ .$$

Let $f(\omega)$ be a non-negative integrable function which is slowly varying and non-increasing for sufficiently large ω. Suppose, as in Theorem 1, that

(6) $$\lim_{\omega\to\infty} K(\omega)/f(\omega) = 0 \quad ,$$

where $K(\omega) = |M(\omega)|^2$, and in addition suppose that $\mathcal{m}(\omega)$ has a positive lower bound for $0 \le \omega \le A$. Then, if $\omega \le \omega'' \le \omega'$ all $\to \infty$ in such a way that $|\omega-\omega'| \le A$, we have

(20) $$\Gamma(\omega,\omega') \;/\; f(\omega'')\mathcal{m}(\omega-\omega') \to 1$$

uniformly, where $\Gamma(\omega,\omega')$ is the covariance (4).

Remark 1. When $\omega = \omega'$, Theorem 2 reduces to Theorem 1.

Remark 2. Both of the remarks made after Theorem 1 apply to Theorem 2 as well.

Remark 3. Of course, Theorem 2 is susceptible to the same kind of extensions as Theorem 1 (see Section 4).

Remark 4. If we choose $\omega'' = \frac{1}{2}(\omega+\omega')$, then (20) asserts that $\Gamma(\omega,\omega')$ is asymptotically the *locally stationary covariance* $f\left[\frac{1}{2}(\omega+\omega')\right]\mathcal{m}(\omega-\omega')$ [7].

Proof of Theorem 2. Since the proof is almost identical with that of Theorem 1, we may be quite brief. We have

$$\Gamma(\omega,\omega') = \int_{-\infty}^{R} M(s)\overline{M}(s+\omega'-\omega)f(\omega-s)ds + \int_{R}^{\infty} M(s)\overline{M}(s+\omega'-\omega)f(\omega-s)ds$$

$$= \Gamma_1(\omega,\omega') + \Gamma_2(\omega,\omega') \qquad ,$$

where $R = R(\omega)$ is a function chosen as in Theorem 1. Applying the dominated convergence theorem to the first integral we get

$$\lim_{\omega\to\infty} \Gamma_1(\omega,\omega')/f(\omega) = \int_{-\infty}^{\infty} M(s)M(s+\omega'-\omega)ds = \mathcal{M}(\omega-\omega') \quad ,$$

where in the left hand side $f(\omega)$ can be replaced by ω'' since $f(\omega)/f(\omega'') \to 1$. Then, applying the elementary inequality $|ab| \leq \frac{1}{2}(|a|^2 + |b|^2)$ to the second integral, we have

$$|\Gamma_2(\omega,\omega')| \leq \frac{1}{2}\int_R^{\infty} \left[|M(s)|^2 + |M(s+\omega'-\omega)|^2\right] f(\omega-s)ds \quad .$$

It follows that $\Gamma_2(\omega,\omega')/f(\omega)$, and hence $\Gamma_2(\omega,\omega')/f(\omega'')$, tends to zero precisely as in Theorem 1, where now $\frac{1}{2}\left[|M(s)|^2 + |M(s+\omega'-\omega)|^2\right]$ plays the role of $K(\omega)$.

6. Two weighted processes

There is another direction in which we can generalize Theorem 1, namely, we can consider the case of two weighted random processes. Thus, let $m_1(t)$ and $m_2(t)$ be two square-integrable weight functions and consider the two processes $y_1(t) = m_1(t)x(t)$ and $y_2(t) = m_2(t)x(t)$, both derived from the same underlying stationary process $x(t)$. Then if $Y_1(\omega)$ and $Y_2(\omega)$ are the Fourier transforms of $y_1(t)$ and $y_2(t)$, which by a previous argument exist with probability one, we have

(21) $$\gamma_{12}(\omega) = EY_1(\omega)\overline{Y}_2(\omega) = \int_{-\infty}^{\infty} M_1(\omega-s)\overline{M}_2(\omega-s)f(s)ds \quad ,$$

where $M_1(\omega)$ and $M_2(\omega)$ are the Fourier transforms of $m_1(t)$ and $m_2(t)$. These considerations suggest the following theorem:

Theorem 3. Let $f(\omega)$ be a non-negative integrable function which is slowly varying and non-increasing for sufficiently large ω. Let $M_1(\omega)$ and $M_2(\omega)$ be square-integrable functions such that

$$\lim_{\omega\to\infty} M_1(\omega)\bar{M}_2(\omega)/f(\omega) = 0.$$

Then

$$\lim_{\omega\to\infty} \gamma_{12}(\omega)/f(\omega) = \int_{-\infty}^{\infty} M_1(\omega)\bar{M}_2(\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} m_1(t)\bar{m}_2(t)dt \quad ,$$

where $\gamma_{12}(\omega)$ is the quantity (21).

Like Theorem 2, the proof of this theorem is essentially the same as that of Theorem 1, where this time we identify $K(\omega)$ with $M_1(\omega)\bar{M}_2(\omega)$. Again the two remarks made after Theorem 1 apply to Theorem 3 as well: moreover, Theorem 3 is susceptible to the same extensions as Theorem 1 (see Section 4). The meaning of Theorem 3 is illustrated by the case where $m_2(t)$ is a translate of $m_1(t)$, i.e. $m_2(t) = m_1(t+\tau)$. We take $m_1(t)$ to be a real-valued data window which peaks about some central value. Suppose that the two data windows overlap only slightly, i.e. suppose τ is such that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} m_1(t)m_1(t+\tau)dt \ll \frac{1}{2\pi}\int_{-\infty}^{\infty} m_1^2(t)dt = 1 \ .$$

Then Theorem 3 assets that $\lim_{\omega\to\infty} \gamma_{12}(\omega)/\gamma(\omega) \ll 1$, where as usual $\gamma(\omega)$ is the convolution (5). Qualitatively, this means that if we take two essentially non-overlapping sections of a stationary process $x(t)$, we shall find that their high-frequency contents are only weakly correlated, even if the two sections both lie <u>well</u> <u>within</u> the correlation distance or memory of $x(t)$, provided, of course, that appropriate conditions are met by $x(t)$ and the data windows involved.

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